

Canonical form of m -by-2-by-2 matrices over a field of characteristic other than two

Genrich Belitskii*

Dept. of Mathematics, Ben-Gurion University of the Negev
Beer-Sheva 84105, Israel, genrich@cs.bgu.ac.il

M. Bershadsky,
Sapir Academic College P.b.
Hof Ashkelon, 79165, Israel, maximb@mail.sapir.ac.il

Vladimir V. Sergeichuk[†]
Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine
sergeich@imath.kiev.ua

Abstract

We give a canonical form of $m \times 2 \times 2$ matrices for equivalence over any field of characteristic not two.

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Each trilinear form $f : U \times V \times W \rightarrow \mathbb{F}$ on vector spaces U , V , and W of dimensions m , n , and q over a field \mathbb{F} is given by the $m \times n \times q$ spatial matrix

$$\mathcal{A} = [a_{ijk}]_{i=1}^m [j=1]_n [k=1]_q, \quad a_{ijk} = f(u_i, v_j, w_k), \quad (1)$$

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with respect to bases $\{u_i\}$ of U , $\{v_j\}$ of V , and $\{w_k\}$ of W . Changing the bases, we can reduce \mathcal{A} by *equivalence transformations*

$$[a_{ijk}] \mapsto [b_{i'j'k'}], \quad b_{i'j'k'} = \sum_{ijk} a_{ijk} r_{ii'} s_{jj'} t_{kk'}, \quad (2)$$

in which $[r_{ii'}]$, $[s_{jj'}]$, and $[t_{kk'}]$ are nonsingular matrices.

Canonical forms of $2 \times 2 \times 2$ matrices for equivalence over the field of complex numbers was given by Ehrenborg [2]. We extend his result to $m \times 2 \times 2$ matrices over any field \mathbb{F} of characteristic not two. Unlike [2], one of our canonical $2 \times 2 \times 2$ matrices depends on a parameter that is determined up to multiplication by any z^2 , $0 \neq z \in \mathbb{F}$.

Note that the canonical form problem for $m \times n \times 3$ matrices for equivalence is hopeless since it contains the problem of classifying pairs of linear operators and, hence, the problem of classifying arbitrary systems of linear operators [1, Theorems 4.5 and 2.1]; such classification problems are called *wild*. The canonical form problem for $m \times n \times 2$ matrices for equivalence was studied in [1, Section 4.1].

We give the spatial matrix (1) by the q -tuple of $m \times n$ matrices

$$\mathcal{A} = \|A_1 \mid \dots \mid A_q\|, \quad A_k = [a_{ijk}]_{ij}. \quad (3)$$

The equivalence transformation (2) can be realized in two steps: by the nonsingular linear substitution

$$C_1 = A_1 t_{11} + \dots + A_q t_{q1}, \dots, C_q = A_1 t_{1q} + \dots + A_q t_{qq}, \quad (4)$$

and then by the simultaneous equivalence transformation

$$\|R^T C_1 S \mid \dots \mid R^T C_q S\|, \quad R = [r_{ii'}], \quad S = [s_{jj'}]. \quad (5)$$

Moreover, two spatial matrices are equivalent if and only if one reduces to the other by nonsingular linear substitutions and simultaneous equivalence transformations of the form (4) and (5). The rank

$$q' = \text{rank}\{A_1, \dots, A_q\}$$

of the matrices A_1, \dots, A_q from (3) in the space of m -by- n matrices is an invariant of \mathcal{A} with respect to equivalence transformations.

Apart from (3), the spatial matrix (1) can be also given by the tuples

$$(\tilde{A}_1, \dots, \tilde{A}_n), \quad (\tilde{\tilde{A}}_1, \dots, \tilde{\tilde{A}}_m) \quad (6)$$

of the matrices $\tilde{A}_j = [a_{ijk}]_{ik}$ and $\tilde{\tilde{A}}_i = [a_{ijk}]_{jk}$, and

$$n' = \text{rank}\{\tilde{A}_1, \dots, \tilde{A}_n\}, \quad m' = \text{rank}\{\tilde{\tilde{A}}_1, \dots, \tilde{\tilde{A}}_m\} \quad (7)$$

are also invariants of \mathcal{A} for equivalence transformations. We say that the spatial matrix \mathcal{A} is *regular* if $m = m'$, $n = n'$, and $q = q'$.

Let \mathcal{A} be irregular. Make the first q' matrices $A_1, \dots, A_{q'}$ in the q -tuple (3) linearly independent and the others $A_{q'+1}, \dots, A_q$ zero by substitutions of the form (4). Then reduce in the same way the tuples (6) of the obtained spatial matrix and get a spatial matrix $\mathcal{B} = [b_{ijk}]$ that is equivalent to \mathcal{A} and whose entries outside of

$$\mathcal{B}' = [b_{ijk}]_{i=1}^{m'} [j=1]^{n'} [k=1]^{q'} \quad (8)$$

are zero. We call \mathcal{B} a *regularized form of \mathcal{A}* and its submatrix (8) a *regular part of \mathcal{A}* .

Two spatial matrices of the same size are equivalent if and only if their regular parts are equivalent [1, Lemma 4.7]. Hence, it suffices to give a canonical form of a regular spatial matrix.

Due to this lemma and the next theorem, the set of all regularized forms whose regular submatrices are (9)–(16) is a set of $m \times 2 \times 2$ canonical matrices for equivalence.

Theorem 1. *Over a field \mathbb{F} of characteristic not two, each regular $m \times n \times q$ matrix \mathcal{A} with $n \leq 2$ and $q \leq 2$ is equivalent to one of the spatial matrices:*

$$\| 1 \| \quad (1 \times 1 \times 1), \quad (9)$$

$$\left\| \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right\| \quad (2 \times 2 \times 1), \quad (10)$$

$$\left\| \begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array} \right\| \quad (2 \times 1 \times 2), \quad (11)$$

$$\| 1 \ 0 \mid 0 \ 1 \| \quad (1 \times 2 \times 2), \quad (12)$$

$$\| I_2 \mid B(a) \| := \left\| \begin{array}{cc|cc} 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 \end{array} \right\| \quad (a \in \mathbb{F}, 2 \times 2 \times 2), \quad (13)$$

$$\left\| \begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\| \quad (3 \times 2 \times 2), \quad (14)$$

$$\left\| \begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\| \quad (3 \times 2 \times 2), \quad (15)$$

$$\left\| \begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\| \quad (4 \times 2 \times 2). \quad (16)$$

These spatial matrices, except for (13), are uniquely determined by \mathcal{A} ; $\|I_2 \mid B(a)\|$ is equivalent to $\|I_2 \mid B(b)\|$ if and only if $a = bz^2$ for some nonzero $z \in \mathbb{F}$.

Proof. Existence. Let \mathcal{A} be $m \times n \times 1$. Since $\mathcal{A} = \|A\|$ is regular, it reduces by elementary transformations (5) to (9) or (10).

Let \mathcal{A} be $1 \times 2 \times 2$. Reduce it to the form $\|1 \ 0 \mid b_1 \ b_2\|$ by transformations (5). Since \mathcal{A} is regular, $b_2 \neq 0$; we make $b_2 = 1$ multiplying the second columns by b_2^{-1} . Adding a multiple of b_2 , make $b_1 = 0$ and obtain (12). Similarly, if \mathcal{A} is $2 \times 1 \times 2$, then it reduces to (11). Note that (11) and (12) can be obtained from the spatial matrix $[a_{ijk}]$ defined in (10) by interchanging its indices.

It remains to consider \mathcal{A} of size $m \times 2 \times 2$ with $m \geq 2$. Since $\mathcal{A} = \|A \mid B\|$ is regular, $A \neq 0$, $B \neq 0$, and the rows of the $m \times 4$ matrix $[A \ B]$ are linearly independent; that is, $m = \text{rank}[A \ B] \leq 4$. Interchanging A and B if necessary, we make

$$\text{rank } A \geq \text{rank } B. \quad (17)$$

If $m = 4$, then $[A \ B]$ is a nonsingular 4×4 matrix and we reduce $\mathcal{A} = \|A \mid B\|$ by row-transformations to the form (16).

If $m = \text{rank}[A \ B] = 3$, then $\text{rank } A = 2$ by (17), we reduce $\mathcal{A} = \|A \mid B\|$ by transformations (5) to the form

$$\left\| \begin{array}{cc|cc} 1 & 0 & b_{11} & 0 \\ 0 & 1 & b_{21} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\|. \quad (18)$$

Replacing B by $B - b_{11}A$, we make $b_{11} = 0$ but spoil $b_{22} = 0$; we fix b_{22} adding the third row. So \mathcal{A} is equivalent to (14) or (15).

Let $m = \text{rank}[A \ B] = 2$. If A is singular, then by (17) $\text{rank } A = \text{rank } B = 1$ and we reduce \mathcal{A} to the form

$$\left\| \begin{array}{cc|cc} 1 & 0 & 0 & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{array} \right\|.$$

Since \mathcal{A} is regular, (b_{21}, b_{22}) and (b_{12}, b_{22}) are nonzero. Because $\text{rank } B = 1$, $b_{21} = 0$ or $b_{12} = 0$. We replace A by the nonsingular $A + B$.

Since A is nonsingular, \mathcal{A} reduces to the form

$$\left\| \begin{array}{cc|cc} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{array} \right\|. \quad (19)$$

Preserving $A = I_2$, we will reduce B by similarity transformations. If $b_{21} = 0$, then we make $b_{21} \neq 0$ using

$$\begin{bmatrix} 1 & 0 \\ -\varepsilon & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \varepsilon & 1 \end{bmatrix}, \quad \varepsilon = 1 \text{ or } \varepsilon = -1.$$

Multiplying the first row of B by b_{21} and its first column by b_{21}^{-1} , we obtain $b_{21} = 1$. Make $b_{11} = -b_{22}$ replacing B by $B - \alpha A = B - \alpha I_2$, where $\alpha := (b_{11} + b_{22})/2$. At last, replace B with

$$\begin{bmatrix} 1 & -b_{11} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ 1 & -b_{11} \end{bmatrix} \begin{bmatrix} 1 & b_{11} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \star \\ 1 & 0 \end{bmatrix}$$

and obtain (13).

Uniqueness. If two spatial matrices among (9)–(16) are equivalent, then they have the same size, and so they are (14) and (15), or $\|I_2 \mid B(a)\|$ and $\|I_2 \mid B(b)\|$. If the spatial matrix (14) is $\|A \mid B\|$, then $\text{rank}(\alpha A + \beta B) \neq 1$ for all $\alpha, \beta \in \mathbb{F}$, hence (14) is not equivalent to (15).

Let $\|I_2 \mid B(a)\|$ be equivalent to $\|I_2 \mid B(b)\|$. By (5) and (4), there is a nonsingular matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ such that the matrices $\alpha I_2 + \beta B(a)$ and $\gamma I_2 + \delta B(a)$ are simultaneously equivalent to I_2 and $B(b)$. Then $\alpha I_2 + \beta B(a)$ is nonsingular and the matrix $(\alpha I_2 + \beta B(a))^{-1}(\gamma I_2 + \delta B(a))$ is similar to $B(b)$. Hence, the matrices

$$\begin{bmatrix} \alpha & -\beta a \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} \gamma & \delta a \\ \delta & \gamma \end{bmatrix} = \begin{bmatrix} \alpha\gamma - \beta\delta a & (\alpha\delta - \beta\gamma)a \\ \alpha\delta - \beta\gamma & \alpha\gamma - \beta\delta a \end{bmatrix}, \quad (\alpha^2 - \beta^2 a)^2 \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix}$$

are similar. Equating their traces and determinants we obtain $\alpha\delta - \beta\gamma = 0$ and $(\alpha\delta - \beta\gamma)^2 a = (\alpha^2 - \beta^2 a)^2 b$. Therefore, $a = bz^2$, where $z = (\alpha\delta - \beta\gamma)/(\alpha^2 - \beta^2 a)$.

Conversely, if $a = bz^2$ and $0 \neq z \in \mathbb{F}$, then $\|I_2 \mid B(b)\|$ is equivalent to $\|I_2 \mid zB(b)\|$, which is equivalent to $\|I_2 \mid B(a)\|$ since

$$\begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & bz \\ z & 0 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & bz^2 \\ 1 & 0 \end{bmatrix}.$$

□

References

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